NPS-53-88-009

NAVAL POSTGRADUATE SCHOOL

Monterey, California

A NEW CONSERVATIVE SCHEME
FOR SOLVING THE TWO BODY PROBLEM

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August 1988

Prepared for:
Naval Postgraduate School
Monterey, CA 93943

MONTEREY, CALIFORNIA

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REPORT DOCUMENTATION PAGE

ORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
URITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited
LASSIFICATION / DOWNGRADING SCHEDULE	
ORMING ORGANIZATION REPORT NUMBER(S) NPS-53-88-009	5 MONITORING ORGANIZATION REPORT NUMBER(S) NPS-53-88-009
ME OF PERFORMING ORGANIZATION Naval Postgraduate School — 6b OFFICE SYMBOL (If applicable) 53	7a NAME OF MONITORING ORGANIZATION
DRESS (City, State, and ZIP Code) Monterey, CA 93943	7b ADDRESS (City, State, and ZIP Code)
ME OF FUNDING / SPONSORING ANIZATION NSWC — 8b OFFICE SYMBOL (If applicable) K-13	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
DRESS (City, State, and ZIP Code) Dahlgren, VA 22448	10 SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO / PROJECT NO / TASK NO. / WORK UNIT ACCESSION NO

LE (Include Security Classification)

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PE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year Month, Day)	15 PAGE COUNT
nical Report	FROM 10/86 TO 6/87	August 1988	

PLEMENTARY NOTATION

COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
LD	GROUP	SUB-GROUP	Two Body Problem, Conservative Scheme

TRACT (Continue on reverse if necessary and identify by block number)

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STRIBUTION / AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
AME OF RESPONSIBLE INDIVIDUAL Neta & Y. Ilan-Lipowski	22b TELEPHONE (Include Area Code) 408-646-2234 — 22c OFFICE SYMBOL 53Nd

DRM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
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ACKNOWLEDGEMENT

The authors would like to thank NSWC for its partial support of this research.

Abstract

The well-known Störmer-Cowell class of linear k-step methods for the solution of second order initial value problems suffer from orbital instability. The solution of a problem describing a uniform motion in a circular orbit, spirals inward. Several modifications were suggested, some require the a-priori knowledge of the frequency. Here we develop a method based on the conservation of energy and momentum. This method overcomes the aforementioned difficulty.

1980 Mathematics Subject Classification 65L05

Key Words: Conservative scheme, Two body problem.

1. Introduction

In this paper we develop a method for the numerical solution of the equations of motion of an object acted upon by several gravitational masses. In general, the motion can be described by a special class of second order differential equations, namely

$$y''(x) = f(x,y(x)) . \tag{1}$$

There exist methods of Runge-Kutta type which tackle this problem directly (Collatz [8, p. 61], de Vogelaere [23], Scraton [20] and others). Also, linear multistep methods of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k'} \beta_j f_{n+j} \tag{2}$$

exist. See, for example, Henrici [11, p. 289], Lambert [13, p. 252] and others. One of the authors has developed hybrid methods for such classes (Neta and Lee [15], Neta [16, 18]).

The direct application of methods of class (2) to problem (1), rather than the application of a conventional linear multistep method to an equivalent first-order system is usually recommended (Ash [1]).

Special methods based on a-priori knowledge of the frequency were developed by Bettis [3], Steifel and Bettis [22], Gautschi [10], Neta and Ford [17], Neta [19], van der Houwen and Sommeijer [12], Lyche [14] and Sommeijer et al [21].

To avoid the use of the frequency other methods were developed satisfying the so-called P-stability. This idea of P-stability was introduced by Lambert and Watson [13]. They showed that such methods are necessarily implicit and cannot have order greater than two. Chawla [5] and Cash [4] independently showed that this order-barrier can be crossed over by considering hybrid two-step methods. Other P-stable methods were developed by Costabile & Costabile [9], Chawla [6], Chawla and Rao [7] and others.

Our idea here is to develop a new method to approximate the solution of the two body problem. This scheme will conserve both the energy per unit mass and the specific angular momentum of the system.

In the next section the method is developed for the perturbation free flight. Numerical experiments with the scheme will be described in the last section.

2. Development of the Method

In this section we describe our new scheme to solve the following system of equations

$$\frac{d^2x}{dt^2} = - \frac{k}{(x^2+y^2)^{3/2}} x , \tag{3}$$

$$\frac{d^2y}{dt^2} = - \frac{k}{(x^2+y^2)^{3/2}} y , \tag{4}$$

where $k = 3.9877 \; 10^{14} \; m^3 \; sec^{-2}$ is the gravitational parameter.

It is well known that Cowell's method gives a numerical solution that spirals inward. Encke's method improves this result but requires more work [2].

It can be easily shown (see e.g., Bate et al. [2]) that the energy and momentum are conserved for a perturbation free flight. The conservation of these quantities will be used to obtain the fast changing variable. It is thus important to rewrite the system in polar coordinates

$$\frac{d^2r}{dt^2} = r\left(\frac{d\theta}{dt}\right)^2 - \frac{k}{r^2} \tag{5}$$

$$\frac{d^2\theta}{dt^2} = - \frac{2}{r} \frac{dr}{dt} \frac{d\theta}{dt} . \tag{6}$$

The radius r does <u>not</u> change much in time and thus we can use any method; here we use a second order Taylor series method. The value of θ corresponding to r will be obtained by integrating the relation (conservation of energy per unit mass)

$$(\frac{dr}{dt})^2 + (r\,\frac{d\theta}{dt})^2 = (\frac{dr}{dt}\Big|_{t=0})^2 + (r(0)\,\frac{d\theta}{dt}\Big|_{t=0})^2 . \qquad (7)$$

To be more specific, our method approximates r using

$$r(t_{i+1}) = r(t_i) + \dot{r}(t_i)(\Delta t) + \frac{1}{2}\ddot{r}(t_i)(\Delta t)^2 \qquad (8)$$

where

$$\dot{r}(t_{i+1}) = \dot{r}(t_i) + \ddot{r}(_i)(\Delta t) \qquad (9)$$

and $\ddot{r}(t)$ is obtained from the differential equation. In a similar fashion we find

$$\theta(t_{i+1}) = \theta(t_i) + \dot{\theta}(t_i)(\Delta t) + \frac{1}{2}\ddot{\theta}(t_i)(\Delta t)^2 \qquad (10)$$

where

$$\dot{\theta}(t_i) = \sqrt{(C + k/r(t_i))^2 - \dot{r}^2(t_i)/r(t_i)} \qquad (11)$$

and $\ddot{\theta}(t_i)$ is obtained by differentiating the above relation.

The results of this method were compared to Taylor series method for both r and θ where now $\dot{\theta}$ is evaluated from an equation similar to (9).

Remark: It should be clear that this idea can be used for any conservative system. One uses the conservation of some quantity to obtain an approximate value for the faster changing component.

3. Numerical Experiments

In this section we solve the equations of motion (3)-(4) combined with the initial values

$$
\begin{aligned}
r(0) &= R + 10^5 , \\
r(0) &= 0 , \qquad\qquad (12) \\
\theta(0) &= 0 , \\
\dot{\theta}(0) &= 8000/r(0) ,
\end{aligned}
$$

where $R = 6.371 \times 10^6$ is the radius of earth.

In the first table we compare the results of our method with Taylor series using $t = 1$ sec. As can be seen in the table, the value of r at the end of each period is nearly constant with our method but decreases as the number of periods increases with Taylor series method.

TABLE 1

At the end of Period	$r\ (10^7\ m)$ Taylor	$r\ (10^7\ m)$ New	Energy per unit mass $(10^8\ m^2/sec^2)$	Specific angular momentum $(10^{11}\ m^2/sec)$
0	.6471	.6471	-.2962	.5277
5	.6320	.6476	-.3019	.5127
10	.6173	.6482	-.3077	.5078
15	.6030	.6487	-.3135	.5029
20	.5890	.6493	-.3194	.4982
30	.5622	.6503	-.3313	.4888
40	.5367	.6513	-.3433	.4798
50	.5123	.6523	-.3557	.4710
60	.4891	.6533	-.3683	.4624
70	.4667	.6543	-.3811	.4540

The energy per unit mass E and the specific angular momentum are given by

$$E = \frac{1}{2}(\dot{r}^2 + (r\dot{\theta})^2) - \frac{k}{r} ,$$

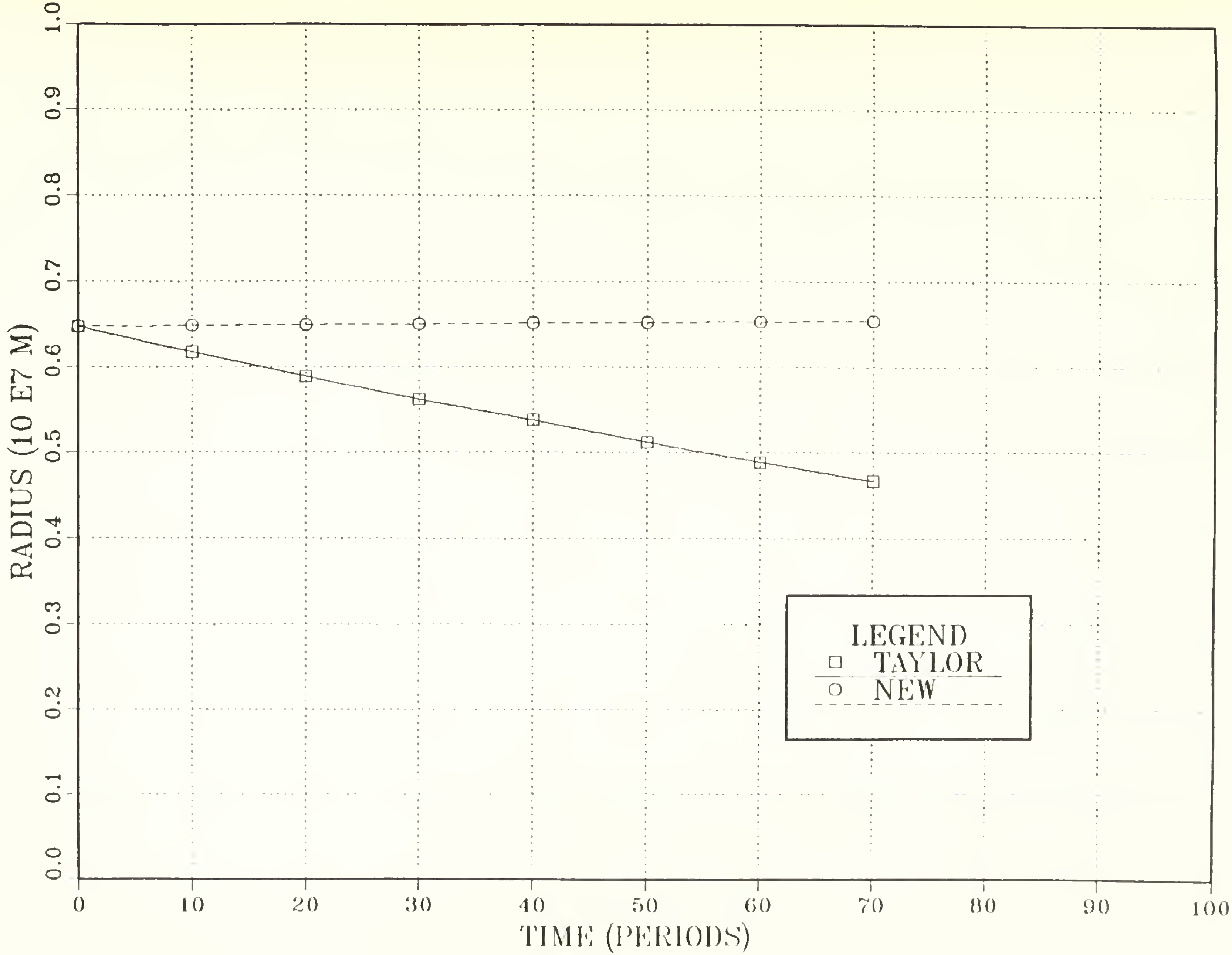
$$P = r^2\dot{\theta} - \dot{r}\theta . \tag{13}$$

In our example both the energy and momentum are conserved.

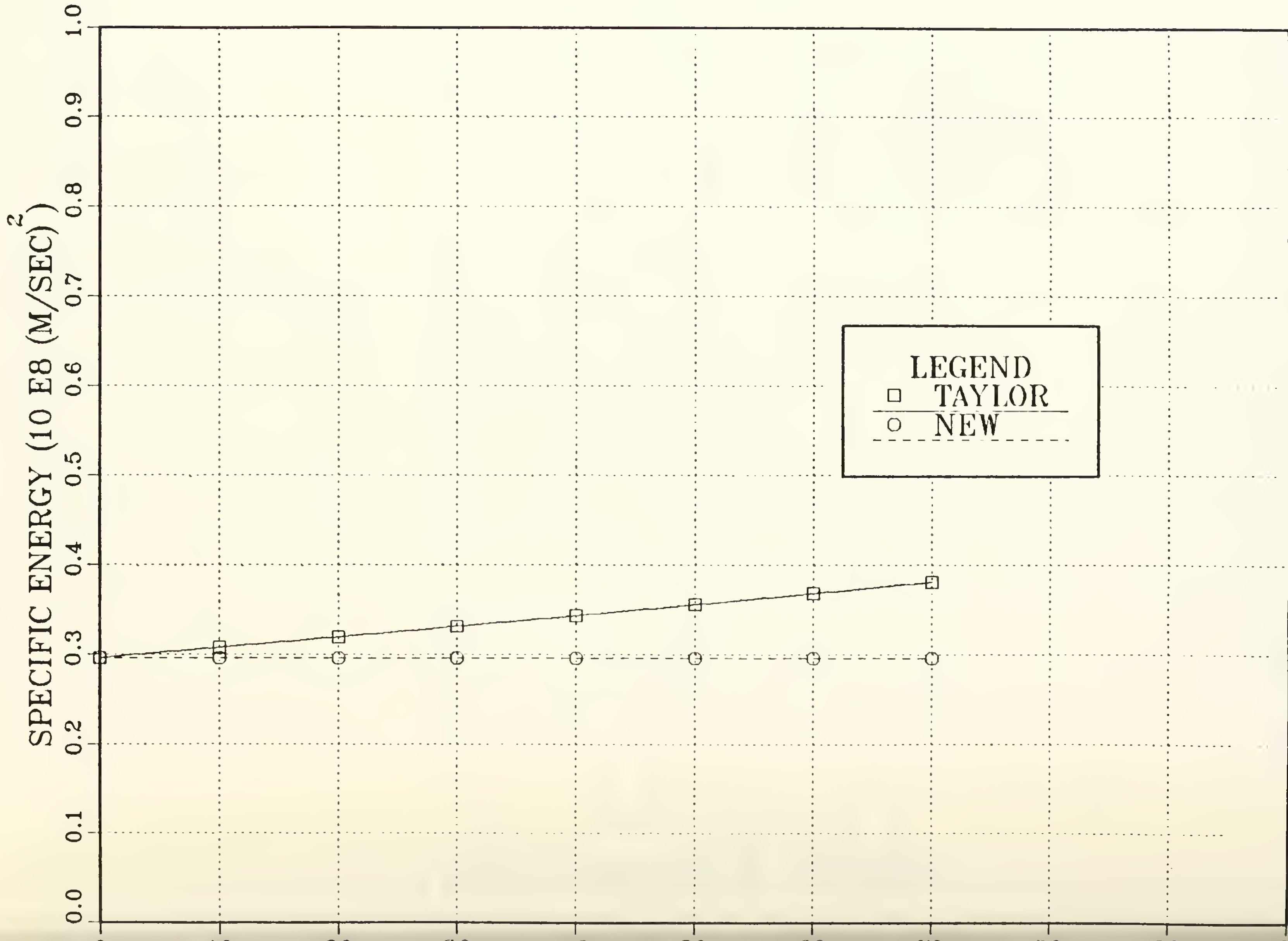
In the following three figures the radius r, the energy per unit mass E and the specific angular momentum P were plotted at the end of every 10 periods. It is clear that the relative change in r values after 70 periods using Taylor series method is more than 25 times larger compared to our new method. The energy and momentum were kept constant by the new scheme whereas the Taylor series method shows a relative loss of 28% in energy and 12% in momentum at the end of 70 periods.

<u>Remark</u>: Note that our method is explicit whereas the P-stable methods suggested by Lambert and Watson [13] are implicit.

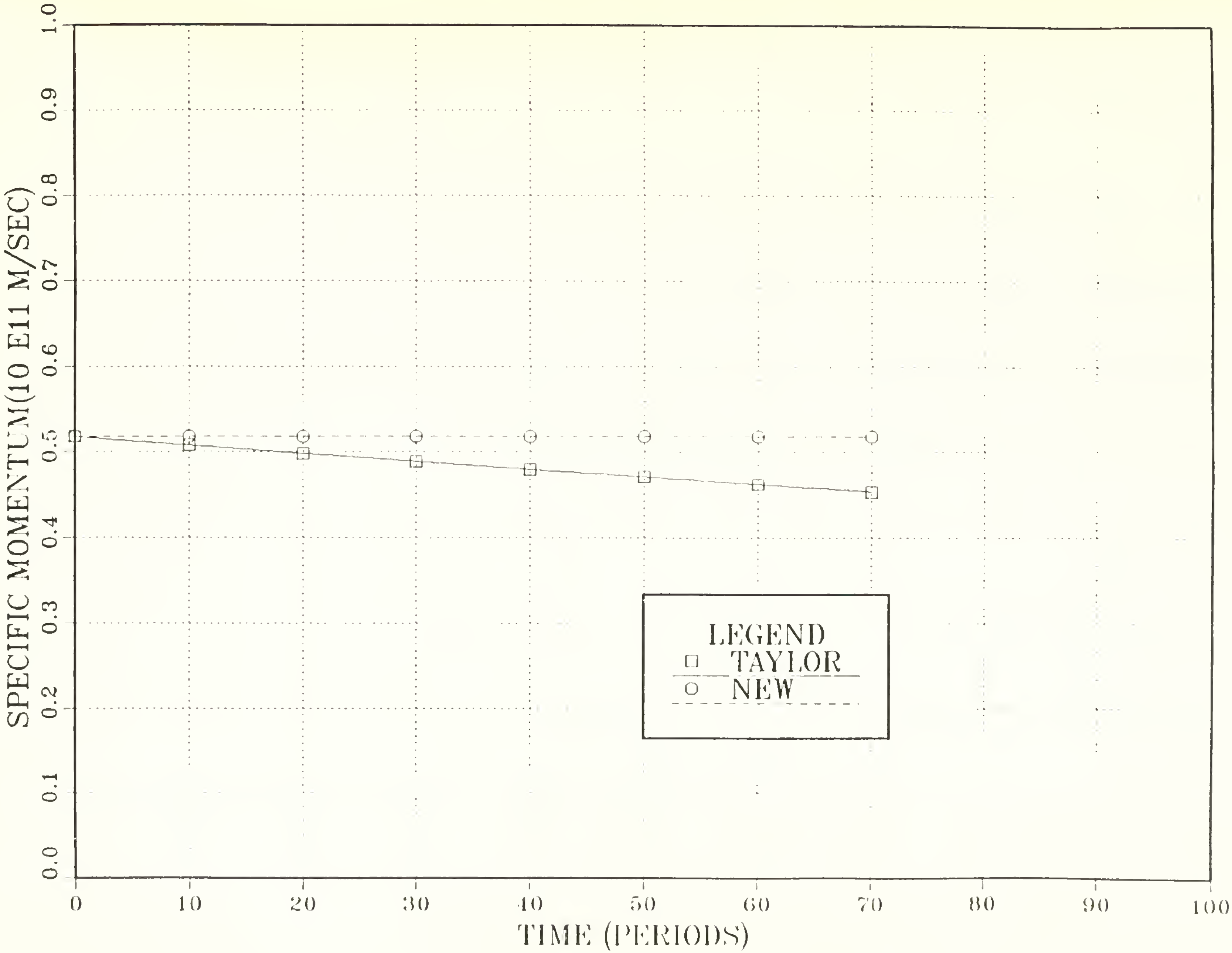
COMPARISON OF RADIUS



COMPARISON OF ENERGY



COMPARISON OF MOMENTUM



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